

# Topological Order in the (2+1)D Compact Lattice Superconductor

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**Abstract.** – We study topological aspects of a compact lattice superconductor, and show that the characteristic energy splitting,  $\Delta$ , between almost degenerate ground states, is simply related to a novel order parameter  $\tilde{W}$ , which is closely related to large Wilson loops. Using Monte Carlo methods, we study the scaling properties of  $\tilde{W}$  close to the deconfining phase transition, and conclude that  $\Delta \sim e^{-L/\xi}$ , where  $L$  is the size of the system, thus giving quantitative support to the vortex tunneling scenario proposed by Wen.

The concept of topological order was first introduced to describe the ground states of the quantum Hall system [1]. Although the present interest in topological order to large extent derives from the quest for exotic non-Fermi liquid states of relevance for the high- $T_c$  superconductors [2–6], the concept is of much more general interest. Both spin liquids [7, 8], believed to be of relevance for frustrated magnets, and ordinary BCS superconductors with dynamical electromagnetism [9, 10], are examples of topologically ordered systems. Two characteristics of topological order are particularly striking; fractionally charged quasiparticles, and a ground state degeneracy depending on the topology of the underlying space, which is lifted by tunneling processes. In the case of the most celebrated example – the Laughlin FQH states – both these properties are well understood [11], but for superconductors, the situation is less clear. It was only in 1990 that Kivelson and Rokhsar pointed out that the quasiparticles are indeed fractional [12], and to our knowledge there has been no quantitative study of ground state degeneracy and splitting.

We improve on this situation by a numerical study of a 2D compact lattice superconductor, using a novel order parameter  $\tilde{W}$ , which is sensitive to vortex tunneling processes. Our results are in *quantitative* agreement with predictions based on topological order, and rules out the possibility of a spontaneously broken discrete symmetry. A compact lattice superconductor captures important properties of real type II superconductors [10], and the 2D case is also of intrinsic interest in the context of effective gauge theories for the high- $T_c$  cuprates [2–6]. Examples of recent proposals are the  $U(1)$  slave-boson theory [2], the nodal liquid and the spinon-chargeon  $Z_2$  theory [3, 4], and compact QED<sub>3</sub> [5].

We use the Villain form of the action [13],

$$S = \frac{J}{2} \sum_{\mathbf{r}\mu} (\nabla_\mu \theta_{\mathbf{r}} + 2\pi n_{\mathbf{r}\mu} - q A_{\mathbf{r}\mu})^2 + \frac{1}{2g} \sum_{\mathbf{r}\mu} (B_{\mathbf{r}\mu} + 2\pi k_{\mathbf{r}\mu})^2, \quad (1)$$

where the charge  $q$  phase field  $\theta_{\mathbf{r}}$  is defined on the sites  $\mathbf{r} = (x_0, x_1, x_2)$ , and the compact vector potential  $A_{\mathbf{r}\mu}$  on the links  $(\mathbf{r}, \mathbf{r} + \mathbf{e}_\mu)$ , of a Euclidian 3D cubic lattice.  $B_{\mathbf{r}\mu} = \epsilon_{\mu\nu\lambda} \nabla_\nu A_{\mathbf{r}\lambda}$  is the dual field strength, which is defined on the links of the dual lattice <sup>(1)</sup>.

We shall in the following concentrate on  $q = 2$  pertinent both to a Cooper paired BCS superconductor and to the more exotic pairing scenarios in effective gauge models [2, 4–6], although we will keep a general  $q > 1$  where appropriate. The phase diagram of the  $q = 2$  model is well established <sup>(2)</sup>: In the strong coupling limit  $J \rightarrow \infty$ , (1) becomes a  $Z_2$  gauge theory, which has a phase transition at finite  $g$ . This transition between the “confined” phase at large  $g$  and the superconducting, or Higgs, phase at small  $g$ , can rigorously be shown to extend to finite  $J$  [14], and is believed to extend all the way to the 3D  $XY$  transition in the  $g \rightarrow 0$  limit. This is confirmed by recent simulations [15]. The line at  $J = 0$  corresponds to compact QED which is confined at all  $g$  because of instanton effects [16].

The topological structure of the theory, which is our present concern, is particularly simple in the  $g \rightarrow 0$  limit of the  $Z_2$  line, where the ground state degeneracy on, say, a torus can be understood in terms of “visons” describing  $Z_2$  magnetic fluxes through the “holes” [4, 7]. At finite  $g$  the ground states are connected by processes where these vortices tunnel around the closed cycles of the torus, leading to a predicted energy splitting  $\Delta \sim e^{-cL/\xi}$  [9]. Close to the phase transition, the vortices begin to proliferate and this simple formula might well break down. Note that in the case of spontaneous breakdown of a discrete symmetry, there would also be tunneling processes connecting different ground states, but with a splitting  $\Delta \sim e^{-cL^2/\xi^2}$  characteristic of domain wall tunneling [9].

To move away from the limiting cases, and to study the region close to the phase transition, which is of particular interest in the models of high- $T_c$  superconductivity referred to above, numerical simulations are mandatory. It will be advantageous to use a dual formulation in terms of flux tubes and monopoles (*i.e.* instantons) [13, 17] and express the partition function for (1) as  $Z = \sum_{\{\mathbf{m}_{\mathbf{r}}, N_{\mathbf{r}}\}} e^{-S}$ , with

$$S = 2\pi^2 J \sum_{\mathbf{r}\mathbf{r}'} \mathbf{m}_{\mathbf{r}} \cdot V_{\mathbf{r}\mathbf{r}'} \mathbf{m}_{\mathbf{r}'} + q^2 \lambda^2 N_{\mathbf{r}} V_{\mathbf{r}\mathbf{r}'} N_{\mathbf{r}'} . \quad (2)$$

Here  $\mathbf{m}_{\mathbf{r}} \in \mathbb{Z}^3$  is the vorticity on the dual links and  $N_{\mathbf{r}} \in \mathbb{Z}$  is the number of monopoles on the dual sites. The flux lines are constrained to start or end on the monopoles only in quanta of  $q$ , *i.e.*,  $\nabla \cdot \mathbf{m}_{\mathbf{r}} = q N_{\mathbf{r}}$ . The interaction is given by  $V_{\mathbf{r}} = L^{-3} \sum_{\mathbf{k}} V_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}$ , with  $V_{\mathbf{k}}^{-1} = \sum_{\nu} 4 \sin^2(k_{\nu}/2) + \lambda^{-2}$ , which for large distances is a screened coulomb interaction  $V(r) = e^{-r/\lambda}/4\pi r$  with a screening length  $\lambda$  given by  $\lambda^{-2} = Jgq^2$ . In the limit  $\lambda \rightarrow 0$  the interaction reduces to an on-site interaction,

$$S = \frac{\Phi_0^2}{2g} \sum_{\mathbf{r}} \mathbf{m}_{\mathbf{r}}^2, \quad (3)$$

where the flux quantum  $\Phi_0 = \frac{2\pi}{q}$ . Since each  $+$ ( $-$ ) monopole has precisely  $q$  outgoing (incoming) flux lines, there are (for  $q > 1$ ) two distinct phases: In the superconducting phase, where

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<sup>(1)</sup>We use a continuum notation whenever convenient, and the lattice version should be obvious.

<sup>(2)</sup>The phase diagram in the  $q = 1$  case is controversial.

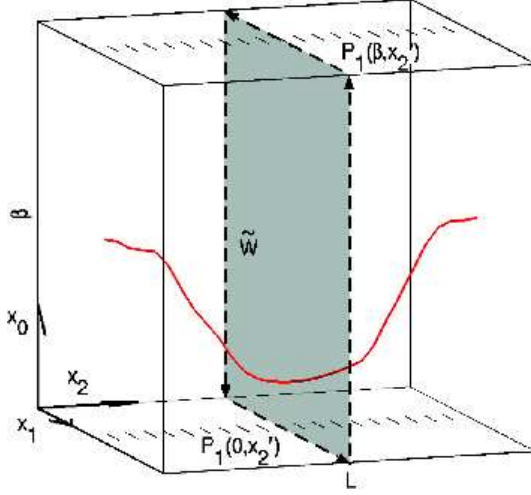


Fig. 1 – Euclidian space-time configuration corresponding to a vortex tunneling around a closed cycle in the  $x_2$  direction on the torus. The operator  $\tilde{W} = \langle \exp(i \oint_C \mathbf{A} \cdot d\mathbf{r}) \rangle$  measures the flux through the loop surrounding the shaded area in the figure. The loop may be decomposed into two vertical Polyakov loops,  $P_0$  and  $P_0^\dagger$ , which cancel each other for periodic boundary conditions in the spatial directions, and two horizontal ones,  $P_1$  and  $P_1^\dagger$ , that differ only on a single twisted link belonging to the indicated column that is twisted by the gauge transformation  $\Omega_{10}$ .

flux lines cost a lot of energy, the monopoles are confined in neutral pairs bound together by  $q$  flux lines. In the other phase, the flux lines condense, connect many monopoles, and form a large connected tangle which percolates through the whole system. The electric properties of these phases are encoded in the behavior of a large Wilson loop  $W = \langle \exp(i \oint_C \mathbf{A} \cdot d\mathbf{r}) \rangle$ , for fractionally charged test particles. In the percolating flux phase, where large flux loops dominate, the flux on any patch the size of a correlation length in a surface spanned by  $\mathcal{C}$ , can be considered as a random variable, thus giving an area law, while in the superconducting phase, the small loops will only contribute close to the edge of  $\mathcal{C}$ , giving a perimeter law [13].

Using periodic boundary conditions on the  $\mathbf{m}$ :s, the total flux for a configuration is just the sum of the  $\mathbf{m}$ :s through a full cross-section  $\mathcal{S}$  of the system in the direction  $\mu$ ,  $\Phi_0 M_\mu = \Phi_0 \sum_{\mathbf{r} \in \mathcal{S}} m_{\mathbf{r}\mu}$ . Clearly  $\tilde{W}_\mu = \langle e^{i\Phi_0 M_\mu} \rangle$  is closely related to large Wilson loops and directly measures the presence of percolating flux, so we expect it to have area law behaviour in the confined phase. In the superconducting phase, however,  $W$  and  $\tilde{W}$  differ – the latter is insensitive to small flux loops and should not obey a perimeter law. For these reasons, we submit that  $\tilde{W}$  is a good (non-local) order parameter for the deconfinement transition, and if the transition is continuous, we expect a finite size scaling relation,

$$\tilde{W}_\mu(\delta, L) = \langle e^{iM_\mu \Phi_0} \rangle = \hat{W}_\pm(L/\xi), \quad \xi \sim |\delta|^{-\nu}, \quad (4)$$

to hold. Here  $\delta$  is a tuning parameter of the transition, *e.g.*,  $\delta = (J - J_c)/J_c$ ,  $\hat{W}_\pm$  is a universal scaling function, and  $\xi$  is the correlation length which diverges as the transition is approached<sup>(3)</sup>.

Physically, configurations having non-zero total flux  $\Phi_0 M_\mu$  around spatial directions of the torus correspond to vortex tunneling events. In order to include these in the partition

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<sup>(3)</sup>For  $q = 1$  this would not give any information, since in this case  $\tilde{W} = 1$  identically.

function we must be careful about boundary conditions. Returning to the original formulation in terms of  $A$  and  $\theta$ , we note that periodic boundary conditions in the  $\mu$  and  $\nu$  directions are compatible only with the total vorticity in the  $\mu\nu$  plane being an integer multiple of  $q$ . We can, however, choose twisted boundary conditions,

$$A_{\mathbf{r}\mu}|_{x_\nu=L} = \Omega_{\mu\nu}(n_{\mu\nu})A_{\mathbf{r}\mu}|_{x_\nu=0}, \quad (5)$$

where  $\Omega_{\mu\nu}$  is a large gauge transformation, that shifts  $A_\mu$  on all links in one particular column in the  $\sigma$  direction, ( $\mu\nu\sigma$  cyclic) by  $2\pi n_{\mu\nu}/q$ , with  $n_{\mu\nu} = 1, \dots, q$ . The twisted links corresponding to  $\Omega_{10}$  are shown in Fig. 1. Acting as an operator on states in the Hilbert space  $\Omega_{\mu\nu}$  changes the value of the Polyakov loop operator  $P_\mu = \exp(i \oint A_\mu)$  (the integration is around a closed cycle on the torus), according to the topological algebra  $\Omega_{\nu\mu}P_\nu - e^{i2\pi n_{\mu\nu}/q}P_\nu\Omega_{\nu\mu} = 0$ . With the boundary conditions (5), the total vorticity in the  $\sigma$  direction is  $n_{\mu\nu} + n_{\nu\mu} \pmod{q}$ .

In order to discuss quantum mechanical states, we now distinguish between the Euclidean time direction  $\mu = 0$ , and the spatial directions  $i = 1, 2$ . For simplicity we also specialize to  $q = 2$  and put  $\Omega_{ij} = \Omega_{ij}(1)$ . Periodicity in space implies even total flux in the 0-direction, and, absent vortices, the Polyakov loop operators,  $P_i$  can be diagonalized, with eigenvalues  $\pm 1$ , so the ground state manifold is spanned by the states  $|f_1, f_2\rangle$ , where  $\Omega_{10}|1, 1\rangle = |0, 1\rangle$  *etc.* Tunneling matrix elements between these states can be extracted from the twisted partition functions,

$$Z_{odd}^i = \text{Tr} [e^{-\beta H} \Omega_{i0}]. \quad (6)$$

The corresponding Euclidean path integral with the boundary condition (5) in the time direction will, in the flux-monopole formulation, amount to summing over configurations with odd total flux through the  $(0i)$  plane. Similarly, the untwisted partition function,  $Z_{ev.}^i$ , corresponds to even total flux. Fig. 1 illustrates that  $Z_{odd}^1$  indeed corresponds to vortex-antivortex pairs tunneling in the 2-direction. In the presence of vortex loops and monopole-antimonopole pairs, the ground states are no longer eigenstates of the loop operators  $P_i(x_0, x_j)$ . Nevertheless, by continuity, the degenerate ground states will persist, and since the effects of flux loops and monopoles in the superconducting phase are local, the tunneling between them is still given solely by the flux lines traversing the whole torus. In a real superconductor with unit charge quasiparticles present, the situation is more complicated, and the ground state is determined by competing tunneling processes [10].

Allowing, for simplicity, tunneling only in the 2-direction and restricting the Hilbert space to the two ground states (which is appropriate at low temperatures), we may write down a tunneling Hamiltonian with elements  $H_{11} = H_{22} = E_0$ ,  $H_{12} = H_{21} = -\Delta$  in the flux basis  $|f_1\rangle$ , which is thus diagonal in the eigenstates  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$  of  $\Omega_{10}$ , with eigenvalues  $E_\pm = E_0 \mp \Delta$ . From the definition of the twisted and untwisted partition function, it now follows,

$$\langle \pm | e^{-\beta H} | \pm \rangle = Z_{ev.} \pm Z_{odd} = e^{-\beta(E_0 \mp \Delta)}, \quad (7)$$

which gives the final formula for the energy splitting,

$$e^{-2\beta\Delta} = \frac{Z_{ev.} - Z_{odd}}{Z_{ev.} + Z_{odd}} = \tilde{W}_\mu, \quad (8)$$

where the last equality follows since the total flux operator  $\Phi_0 M_\mu$  takes the values 0 and  $\pi$  for  $q = 2$ . Not only does this result allow the gap  $\Delta$  to be explicitly calculated from  $\tilde{W}$ . It also implies, via Eq. (4), a scaling relation,  $\Delta = \hat{\Delta}_\pm(L/\xi)/L$ . Expressed in the original

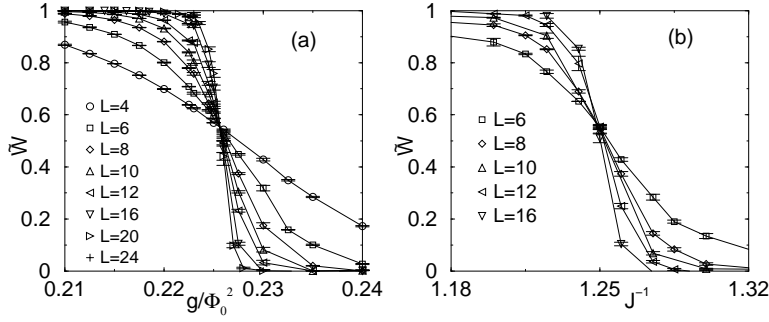


Fig. 2 –  $\tilde{W}$  as function of  $g$  for the  $\lambda \rightarrow 0$  case (a), and of  $J^{-1}$  for  $\lambda = 0.5$  (b). The different curves cross right at the phase transition, where  $\tilde{W}$  becomes independent of the system size, according to Eq. (4).

variables,  $\tilde{W}$  takes the form  $\tilde{W} = \langle P_1(0, x_2) P_1^\dagger(\beta, x_2) \rangle$ , *c.f.* Fig. 1 <sup>(4)</sup>. The generalization to the full 4 state system is straightforward – the splitting pattern is now  $(-2\Delta, 0, 0, 2\Delta)$ , while the relation (8) between  $\Delta$  and  $\tilde{W}$  still holds. It is also possible to generalize to higher  $q > 1$  and other topologies.

Deep into the superconducting phase we can estimate  $\tilde{W}$  by assuming that straight, and statistically independent, paths dominate the partition sum. An easy calculation using (3) gives  $\Delta_0 = (L/a^2)e^{-cL/a}$ , where  $a$  is the lattice spacing, and  $c$  is the line tension of the flux lines ( $c = \Phi_0^2/2g$  for  $\lambda \ll a$  and  $J\pi \ln(\lambda/a)$  for  $\lambda \gg a$ ). To extend this analysis away from the small  $\lambda$  and small  $g$  part of the phase diagram, and especially to the region close to the phase transition, we match the splitting  $\Delta_0$  deep in the phase to the scaling expression given in Eq. (4), leading to

$$\Delta = \frac{L}{\xi^2} e^{-c'L/\xi}, \quad (9)$$

for  $\xi \lesssim L$ , where  $\xi \sim |\delta|^{-\nu}$  is now a physical correlation length. Thus the energy splitting gets renormalized and eventually closes as the transition is approached. The exponentially small splitting has been predicted by Wen, whereas here we obtain also a nontrivial prefactor.

Equation (8) enables us to explicitly calculate the splitting  $\Delta$  numerically using Monte Carlo simulations. We simulate the system in the flux line – monopole representation given by Eq. (2) or (3), using a variant of a recently described worm cluster update Monte Carlo algorithm [18], properly adapted to long range interactions and the existence of magnetic monopoles. This algorithm naturally includes global moves which change  $M_\mu$  (*i.e.*, we allow twists in the boundary conditions (5)) that are necessary for the calculation of  $\tilde{W}$ , whereas in a conventional Metropolis algorithm the acceptance ratio for such moves becomes exponentially small with increasing lattice size. For convenience we allow twists not only in the time direction but also in space, *i.e.*, we simulate the system in a grand canonical ensemble. We have checked that this does not change our results. The details of the numerical methods will be described elsewhere [19].

We have carried out simulations for constant  $\lambda = 0, 0.5, 1, 2$ , varying  $J$  in each simulation. For a given  $\lambda$ , the critical value  $J_c$  (or  $g_c$  in case  $\lambda = 0$ ) of the deconfinement transition is

<sup>(4)</sup>This operator differs from 1 only at the one particular link where the  $A$  field is twisted. This explains, in terms of the original variables, why  $\tilde{W}$  has no perimeter law.

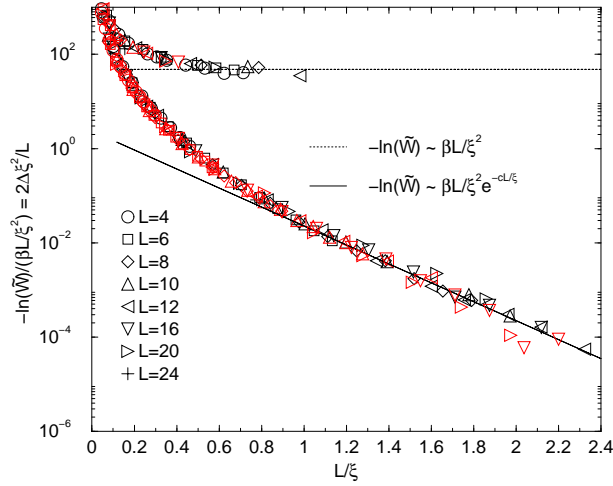


Fig. 3 – Test of Eq. (9) for  $\lambda = 0$ . When plotted against  $L/\xi$ , the combination  $2\Delta\xi^2/L$  falls onto two separate branches. The data for  $g < g_c$  shows an exponential dependence (indicated by the solid straight line) in agreement with Eq. (9). The data for  $g > g_c$  shows an area law indicated by the horizontal dotted line. Black symbols are for  $\beta = L$  and red (gray) symbols are for  $\beta = 2L$ .

located using the nonlocal order parameter  $\tilde{W}$ , as shown in Fig. 2. According to Eq. (4),  $\tilde{W}$  for different system sizes should cross right at the transition. In the topologically ordered phase  $\tilde{W}$  tends to one, implying that  $\Delta \rightarrow 0$  and the states become close to degenerate. The exponential dependence of  $\Delta$  on  $L$  in Eq. (9) is obtained in Fig. 3 for  $\lambda = 0$  and in Fig. 4 for  $\lambda = 0.5$ . We plot  $2\Delta\xi^2/L$  vs  $L/\xi$ , where the correlation length is given by  $\xi = A|\delta|^{-\nu}$ . The correlation length exponent  $\nu$  is adjusted until the data collapse onto two branches (for positive and negative  $\delta$ ), which happens for  $\nu = 0.63$ , consistent with the 3D Ising critical behavior expected for  $q = 2$ . The proportionality constant  $A$  is fixed by Eq. (9). The straight

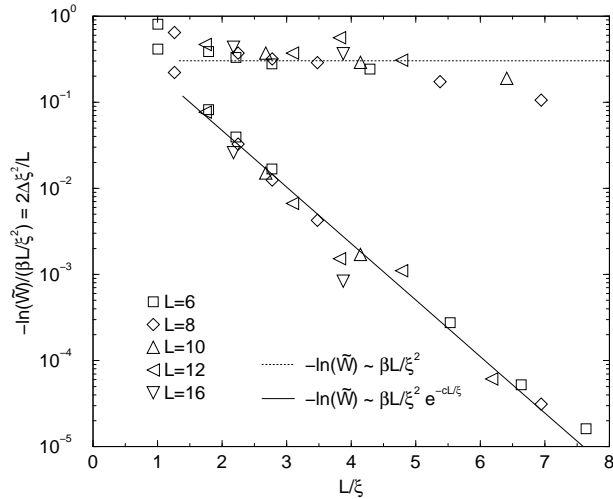


Fig. 4 – Same as Fig. 3, but for  $\lambda = 0.5$ .

line in the lin-log plot indeed demonstrates the exponential scaling in the topologically ordered phase, whereas an area law indicated by the horizontal line is obtained in the confined phase. Note that a  $e^{-cL^2/\xi^2}$  behavior characteristic of a spontaneously broken discrete symmetry is excluded. The results for  $\lambda = 1$  and 2 are very similar. In principle, the simulations done for  $\beta = L$  cannot rule out a weak dependence on  $\beta/L$  in Eq. (9). Therefore we have also calculated  $\Delta$  from anisotropic systems with  $\beta = 2L$ , see Fig. 3, showing that any such residual dependence on  $\beta/L$  must be extremely weak.

In summary, we have shown that the ground state energy splitting  $\Delta$ , between the almost degenerate ground states in the topologically ordered phase of a lattice superconductor is simply related to a novel nonlocal order parameter  $\tilde{W}$  of the deconfining transition. Using a Monte Carlo algorithm we calculated  $\tilde{W}$ , demonstrated that it is a good order parameter for the phase transition, and established that  $\Delta$  has an exponential dependence on system size in full agreement with the vortex tunneling mechanism proposed by Wen. This suggests that the lattice superconductor shows the characteristics of topological order all the way to the phase transition. Let us finally remark that the methods used in this paper should be applicable also to a model with quenched impurities.

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